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A NOTE ON PRIMITIVE MATRICES*

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P-344 B

12 November 1952

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*Research undertaken under contract between the Cowles Commission for Research in Economics and The RAND Corporation

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Suppose that A is a square matrix consisting of nonnegative elements. In certain considerations it is important to know when all the elements of some power of A are strictly positive. Frobenius [2][2] gave a very simple necessary and sufficient condition for this to happen. In this note we give a simple proof of this result. Our proof is algebraic in nature and avoids the use of the convergence of powers of a matrix.

All matrices considered here will have real elements. For two such matrices (not necessarily square) $B = (b_{ij})$, $C = (c_{ij})$ we define

$$B \geqq C \quad \text{if} \quad b_{ij} \geqq c_{ij} \quad \text{for each } i, j.$$

$$B \geq C \quad \text{if} \quad B \geqq C \text{ but } B \neq C$$

$$B > C \quad \text{if} \quad b_{ij} > c_{ij} \quad \text{for each } i, j.$$

A square matrix $A \geqq 0$ (A is then called nonnegative) is said to be indecomposable if for no permutation matrix P does

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$ where the A_{ii} are square submatrices.

The fundamental result about nonnegative, indecomposable matrices is due to Frobenius [2]; this, and other, results have recently been rederived and extended in a greatly simplified manner by Wielandt [3] and Debreu and Herstein [1]. It is

THEOREM. Let $A \geqq 0$ be an indecomposable matrix. Then A has a positive characteristic root r such that

1. r is a simple root.

2. to r can be associated a characteristic vector $x > 0$.
3. if α is any other characteristic root of A, $|\alpha| \leq r$.

If $A > 0$ then 3. can be sharpened to $|\alpha| < r$ for all characteristic roots $\alpha \neq r$ of A.

If $A \geq 0$ is indecomposable and if A has no characteristic root other than r of maximal absolute value then A is said to be primitive.

In this paper we prove the

THEOREM* (Frobenius). Let $A \geq 0$. Then $A^m > 0$ for some integer $m > 0$ if and only if A is primitive.

Suppose that $A^m > 0$. Then A must be indecomposable; for if

$$PAP^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \quad \text{then } PA^mP^{-1} = \begin{pmatrix} B^m & C_m \\ 0 & D^m \end{pmatrix} \quad \text{contradicting } A^m > 0.$$

Now suppose that r and $re^{i\varphi} \neq r$ are characteristic roots of A of maximal absolute value. Then A^m, A^{m+1} are both positive and have r^m, $r^m e^{im\varphi}$, and r^{m+1}, $r^{m+1}e^{i(m+1)\varphi}$ respectively as roots of maximal absolute value. Since the largest root of a positive matrix is simple and is actually greater than any other root in absolute value. We must have $r^m e^{im\varphi} = r^m$, $r^{m+1}e^{i(m+1)\varphi} = r^{m+1}$, whence $e^{i\varphi} = 1$, a contradiction.

There remains but to show that if A is primitive then $A^m > 0$ for a suitable integer $m > 0$. This will be proved as a consequence of the following few lemmas, which by themselves are of some interest.

Lemma 1. If A is primitive then A^m is primitive for every positive integer m.

Proof. Since r is a simple root of A and is the only root of A of absolute value r, r^m is a simple root of A^m and is the only root of A^m of absolute value r^m. So we need but show that A^m is indecomposable for every integer

$m > 0$. Suppose that for some s A^s is not indecomposable; we can then assume that $A^s = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Now $Ax = rx$ for $x > 0$, so $A^s x = r^s x$; partition x according to the partitioning of A^s and we have $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

That is $Dx_2 = r^s x_2$, and since x_2 is positive, r^s is a characteristic root of D. Since the transpose, A', of A is also indecomposable, we have $A'Y = rY$ for $Y > 0$. Partitioning as above we obtain that r^s is a characteristic root of B', and so of B. Being a characteristic root of both B and D r^s must be a multiple root of A^s, which is a contradiction. The lemma is thereby proved.

Lemma 2. (Wielandt). _Let ϵ be any positive number. Suppose $A \geq 0$ is an nxn indecomposable matrix. Then $(\epsilon I+A)^{n-1} > 0$ where I is the identity matrix._

Proof. It clearly suffices to show that for any vector x, $x \geq 0$, $(\epsilon I+A)^{n-1} x > 0$. Let

$$x_\nu = (\epsilon I+A)^{\nu-1} x. \quad \text{Then} \quad x_{\nu+1} = \epsilon x_\nu + Ax_\nu .$$

Hence a zero component can occur in $x_{\nu+1}$ only where a zero component already occured in x_ν. However, not every such zero component can be preserved in $x_{\nu+1}$. For if so, by a suitable reordering of the coordinates,

$$x_\nu = \begin{pmatrix} p \\ 0 \end{pmatrix}, \; p > 0, \text{ whence } \; x_{\nu+1} = \epsilon \begin{pmatrix} p \\ 0 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

from which it follows that $A_{21}p = 0$. This together with $p > 0$ forces $A_{21} = 0$, violating the indecomposability of A. So each application of $\epsilon I + A$ to x decreases the number of zero coordinates by at least one. Hence $(\epsilon I+A)^{n-1}x > 0$.

As an easy consequence of Lemma 2 we obtain

Lemma 3. <u>If</u> $A = (a_{ij})$ <u>is indecomposable and</u> $a_{ii} > 0$ <u>for each</u> i <u>then</u> $A^{n-1} > 0$.

For let ϵ be chosen satisfying $0 < \epsilon < \min_i a_{ii}$. Then $A = \epsilon I + B$ where $B \geq 0$ is indecomposable. Lemma 2 then yields $A^{n-1} > 0$.

Let $A^m = (a_{ij}^{(m)})$. Then we have

Lemma 4. <u>Let</u> $A \geq 0$ <u>be indecomposable. Then for any</u> i,j <u>we can find an</u> $m = m(i,j) > 0$ <u>so that</u> $a_{ij}^{(m)} > 0$.

<u>Proof</u>. Consider first the case $i \neq j$. Since $(I+A)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \ldots + I > 0$ by Lemma 2, $a_{ij}^{(m)} > 0$ for some $m \leq n-1$. Now suppose $i = j$. Since A is indecomposable, no column of zeros can occur in A. So there is a k with $a_{ki} > 0$. If $k = i$ then $a_{ii}^{(m)} > 0$ for all m trivially. If, on the other hand, $k \neq i$, then $a_{ik}^{(m)} > 0$ for some m, and since $a_{ii}^{(m+1)} = \sum_r a_{ir}^{(m)} a_{ri} \geq a_{ik}^{(m)} a_{ki} > 0$ the lemma is proved.

We are now in position to complete the proof of Theorem*. Let A be primitive. Pick m_1 so that in A^{m_1}, $a_{11}^{(m_1)} > 0$. Let $A_1 = A^{m_1} = (a_{ij}(1))$. By Lemma 1 A_1 is primitive, so there is an m_2 such that in $A_1^{m_2}$, $a_{22}^{(m_2)}(1) > 0$. Since $a_{11}(1) = a_{11}^{(m_1)} > 0$, $a_{11}^{(m_2)}(1) > 0$. Let $A_2 = A_1^{m_2}$. Continuing in this way we arrive at an $A_n = A^{m_1 m_2 \cdots m_n}$ which is primitive and whose diagonal elements are all positive. By Lemma 3 $A_n^t > 0$ for some t, hence $A^m > 0$ for some suitably chosen integer m.

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FOOTNOTES

1. This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under subcontract to the RAND Corporation.

2. Numbers in square brackets refer to the bibliography at the end of this paper.

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